October Math Gems

PROBLEM OF THE WEEK 5

§1 Problems

Problem 1.1. The expression $\frac{\sin\frac{\theta}{2}+\sin\theta}{1+\cos\frac{\theta}{2}+\cos\theta} equals$

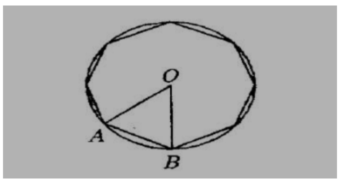
answer. note that $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$
$1+\cos\theta = 2\cos^2\frac{\theta}{2}$

$$\frac{\sin\frac{\theta}{2}+\sin\theta}{1+\cos\frac{\theta}{2}+\cos\theta} = \frac{\sin\frac{\theta}{2}+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1+\cos\frac{\theta}{2}+2\cos^2\frac{\theta}{2}-1} = \frac{\sin\frac{\theta}{2}(1+2\cos\frac{\theta}{2})}{\cos\frac{\theta}{2}+2\cos^2\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}(1+2\cos\frac{\theta}{2})}{\cos\frac{\theta}{2}(1+2\cos\frac{\theta}{2})} = \tan\frac{\theta}{2}$$ □

Problem 1.2. The minimum value of $\frac{1}{3\sin\theta-4\cos\theta+7}$ is

answer. We have, $-5 \le 3\sin\theta - 4\cos\theta \le 5$ for all θ ==> $2 \le 3\sin\theta - 4\cos\theta \le 12$ for all θ ==> $\frac{1}{12} \le \frac{1}{3\sin\theta-4\cos\theta} \le \frac{1}{2}$ for all θ □

Problem 1.3. Find the area of the regular octagon inscribed in a circle of radius r



answer. The measure of angle $AOB = \frac{360}{8} = 45\circ$, The area of triangle $AOB = \frac{1}{2}r^2\sin 45 = \frac{1}{2\sqrt{2}r^2}$
The area of octagon = 8*area of triangle $AOB = 2\sqrt{2}r^2$ □

Problem 1.4. A quadratic polynomial $p(x)$ with real coefficients and leading coefficient 1 is called disrespectful if the equation $p(p(x)) = 0$ is satisfied by exactly three real numbers. Among all the disrespectful quadratic polynomials, there is a unique such polynomial $\tilde{p}(x)$ for which the sum of the roots is maximized. What is $\tilde{p}(1)$?

Solution. Answer: $\frac{5}{16}$
Let the roots be a, b; one of the roots needs to be the minimum value of the quadratic. without loss of generality it's a:

$$a = p\left(\frac{a+b}{2}\right) = \left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right) = -\frac{(a-b)^2}{4}.$$

This is equivalent to

$$1 - 2(a+b) = (b-a-1)^2 \ge 0$$

hence $a+b \leq \frac{1}{2}$, with equality if and only if $b-a=1$, which implies $a=-\frac{1}{4}, b=\frac{3}{4}$. So

$$\tilde{p}(1)=(1-a)(1-b)=\boxed{\frac{5}{16}}.$$

□

Problem 1.5. Prove that the following expression has no real solutions

$$\sqrt{2-x^2}+\sqrt[3]{3-x^3}=0$$

Solution. For this expression to be equal to 0, two conditions must be satisfied. First,

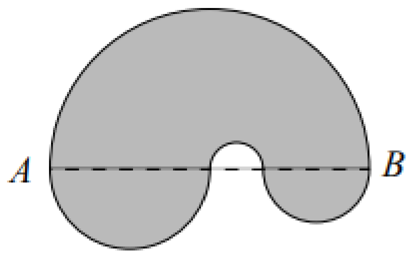
$$\sqrt{2-x^2} \geq 0$$

and

$$\sqrt[3]{3-x^3} \leq 0$$

However, this would require that $x \leq \sqrt{2}$ and $x \geq \sqrt[3]{3}$. Satisfying both conditions simultaneously is not possible. Therefore, the given expression has no real solutions. □

Problem 1.6. The boundary of the shaded figure consists of four semicircular arcs whose radii are all different. The centre of each arc lies on the line AB , which is 10 cm long. What is the length of the perimeter of the figure?



Solution. The centre of the large semicircular arc lies on AB, so we know that AB is a diameter of the large semicircle. But AB is 10 cm long, so the radius of the large semicircle is 5 cm.
Let the radii of the other three semicircles be r_1 cm, r_2 cm and r_3 cm. The centres of these arcs also lie on AB, so the sum of their diameters is equal to the length of AB. It follows that $2r_1+2r_2+2r_3=10$ and hence $r_1+r_2+r_3=5$.
Now the lengths, in cm, of the semicircular arcs are 5π, πr_1, πr_2 and πr_3. Therefore the perimeter of the figure has length, in cm,

$$\begin{aligned} 5\pi+\pi r_1+\pi r_2+\pi r_3 &= \pi(5+r_1+r_2+r_3) \\ &= \pi(5+5) \\ &= 10\pi. \end{aligned}$$

Hence the perimeter of the figure has length 10π cm. □

Problem 1.7. Solve the equation

$$\sin(x)-\sin(x)\cos(x)+\cos(x)=1$$

Solution. $\boxed{\textbf{Answer: } 2k\pi,\ \dfrac{\pi}{2}+2k\pi,\ k\in\mathbb{Z}}$
First, we can subtract 1 from both sides and take $\sin(x)$ as a common factor from the first two terms. We get

$$\sin(x)(1-\cos(x))-(1-\cos(x))=0$$

Taking $1-\cos(x)$ as a common factor, we get

$$(1-\cos(x))(\sin(x)-1)=0$$

This gives two possibilities: $\cos(x)=1$ or $\sin(x)=1$ The solution for these two possibilities are $x=2k\pi, k\in\mathbb{Z}$ and $x=\frac{\pi}{2}+2k\pi, k\in\mathbb{Z}$ respectively. Hence, the solutions for x are

$$x=2k\pi,\frac{\pi}{2}+2k\pi,k\in\mathbb{Z}$$

□

Problem 1.8. If $x, y\in\mathbb{R}$, what is the minimum value of

$$B(x,y)=x^2+2xy+2y^2+4y+2x-2017$$

and for what values of x and y is it achieved?

Solution. $\boxed{\textbf{Answer: -2019, 0, and -1}}$
First, we can rewrite the polynomial as

$$(x^2+y^2+1+2xy+2y+2x)+(y^2+2y+1)-2019$$

Then, we can factorize it to be

$$(x+y+1)^2+(y+1)^2-2019$$

Since, both terms are always greater than or equal to 0 (becuase they are both squared), we can deduce that

$$(x+y+1)^2+(y+1)^2-2019\geq -2019$$

This can also be written as

$$(x+y+1)^2+(y+1)^2\geq 0$$

So, the minimum value is attained when

$$\begin{cases}(x+y+1)^2=0\\(y+1)=0\end{cases}$$

Therefore, the minimum value, which is -2019 is achieved when $y=-1$ and $x=0$ □

Problem 1.9. What is the value of

$$\frac{1}{10^{-9}+1}+\frac{1}{10^{-8}+1}+...+\frac{1}{10^{8}+1}+\frac{1}{10^{9}+1}$$

Solution. $\boxed{\textbf{Answer: 9.5}}$
First, we can write the each expression is in the form of $a(k) = \frac{1}{10^k+1}$. So, we can try to see what the value of $a(k) + a(-k)$ is. This is written as

$$\frac{1}{10^k+1} + \frac{1}{10^{-k}+1}$$

We can then add both fractions and simplify to get

$$a(k) + a(-k) = \frac{10^k + 10^{-k} + 2}{(10^{-k}+1)(10^k+1)}$$

we can then multiply both terms of the denominator and get that it will be exactly the same value as the numerator. Therefore, we deduce that $a(k) + (-k) = 1$. Hence, the expression is equal to $9 + a(0)$. Since, $a(0) = 0.5$, we arrive to our final answer of 9.5 ☐

Problem 1.10. Simplify the given expression

$$10\sqrt{10\sqrt{10\sqrt{10\sqrt{10\sqrt{...}}}}}$$

Solution. $\boxed{\textbf{Answer: 100}}$
Let this expression be equal to x. Therefore,

$$x^2 = 100 \times 10\sqrt{10\sqrt{10\sqrt{10\sqrt{10\sqrt{...}}}}}$$

We can substitute with x for term $10\sqrt{10\sqrt{10\sqrt{10\sqrt{10\sqrt{...}}}}}$. So, we get that

$$x^2 = 100x$$

Therefore, $x = 100$ ☐

Problem 1.11. If 6 people are seated around a circular table, what is the chance that two particular people always seated together ?

Solution. $\boxed{\textbf{Answer: } \frac{2}{5}}$
6 people can be seated around a table in $(n-1)!$ ways, which is 5! ways. If we consider the 2 people to be one person, then there is 4! ways for the 5 people to be seated around the table and 2! ways for the 2 people to be seated. Therefore, the probability that they are seated together is

$$\frac{4! \times 2!}{5!} = \frac{2}{5}$$

☐

Problem 1.12. An ice-cream shop let you choose 2 out of 7 flavors at once and 1 out of 3 toppings. How many different icecreams can you make?

Problem 1.13. In $\triangle ABC$ let point D be the foot of the altitude from A to $\overline{BC}$. Suppose that $\angle A = 90^\circ$, $AB - AC = 5$, and $BD - CD = 7$. Find the area of $\triangle ABC$

Solution. $\boxed{\textbf{Answer: 150}}$
Let $AC = x$ and $AD = h$. Then $AB^2 = (x+5)^2 = BD^2 + h^2$ and $AC^2 = x^2 = CD^2 + h^2$, implying

$$(AB^2 - AC^2) = (BD^2 + h^2 - CD^2 - h^2)$$
$$\therefore (x+5)^2 - x^2 = (BD - CD)(BD + CD).$$

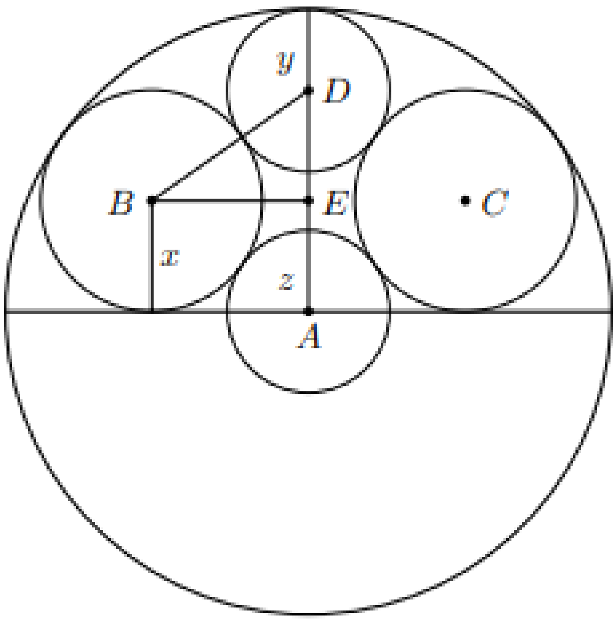
Thus, $10x + 25 = 7BC$ and, from the Pythagorean Theorem,

$$\frac{(10x+25)^2}{49} = (x+5)^2 + x^2.$$

This equation reduces to $x^2 + 5x - 300 = 0$, whose positive solution is $x = 15$. Hence, $AC = 15$, $AB = 20$, and $BC = 25$, implying $AD = h = 12$. The requested area is $\dfrac{AD \cdot BC}{2} = \dfrac{12 \cdot 25}{2} = 150$. $\square$

Problem 1.14. Let S be a sphere with radius 2. There are 8 congruent spheres whose centers are at the vertices of a cube, each has radius x, each is externally tangent to 3 of the other 7 spheres with radius x, and each is internally tangent to S. There is a sphere with radius y that is the smallest sphere internally tangent to S and externally tangent to 4 spheres with radius x. There is a sphere with radius z centered at the center of S that is externally tangent to all 8 of the spheres with radius x. Find $18x + 5y + 4z$.

Solution. $\boxed{\textbf{Answer: 18}}$
The centers of the 8 spheres with radius x are at the vertices of a cube with side length $2x$. Each of these vertices is a distance $x\sqrt{3}$ from the center of S, so it follows that the radius of S is $x(\sqrt{3}+1) = 2$, so $x = \frac{2}{\sqrt{3}+1} = \sqrt{3} - 1$. Let A be the center of S, let B and C be centers of 2 of the spheres with radius x that lie diagonally across a face of the cube, so that $BC = 2\sqrt{2}x$, and let D be the center of the sphere with radius y tangent to the spheres centered at B and C. Let E be the projection of B onto $\overline{AD}$, as shown.



The radius of S is $2 = AE + DE + y = x + \sqrt{BD^2 - BE^2} + y = x + y + \sqrt{(x+y)^2 - 2x^2}$. Then $(2 - (x+y))^2 = (x+y)^2 - 2x^2$, which simplifies to $y = 1 - x + \frac{x^2}{2} = 4 - 2\sqrt{3}$. Also,

$AB^2 = AE^2 + BE^2$, so $(x+z)^2 = x^2 + 2x^2$, which simplifies to $z = 4 - 2\sqrt{3} = y$.The requested expression is $18(\sqrt{3} - 1) + 9(4 - 2\sqrt{3}) = 18$. □

Problem 1.15. Find the number of rearrangements of the nine letters AAABBBCCC where no three consecutive letters are the same. For example, count AABBCCABC and ACABBCCAB but not ABABCCCBA.

Solution. $\boxed{\textbf{Answer: 1314}}$
There are $\binom{9}{3,3,3} = \frac{9!}{3!3!3!} = 1680$ permutations of the nine letters. Let X be the set of permutations where three As appear together, Y be the set of permutations where three Bs appear together, and Z be the set of permutations where three Cs appear together. Then the Inclusion/Exclusion Principle gives the size of the union of these three sets as

$$\begin{aligned}
|X \cup Y \cup Z| &= (|X| + |Y| + |Z|) - (|X \cap Y| + |X \cap Z| + |Y \cap Z|) + |X \cap Y \cap Z| \\
&= 3 \cdot \binom{7}{3,3,1} - 3 \cdot \binom{5}{3,3,1} + 3! \\
&= 3 \cdot 140 - 3 \cdot 20 + 6 \\
&= 366.
\end{aligned}$$

The requested number of permutations is then $1680 - 366 = 1314$. □

Problem 1.16. Starting at $12:00:00$ AM on January 1, 2022, after 13! seconds it will be y years (including leap years) and d days later, where $d < 365$. Find $y + d$.

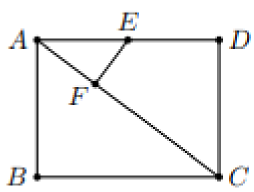
Solution. $\boxed{\textbf{Answer: 317}}$
Divide 13! by $60 = 5 \cdot 12$ to get the number of minutes, by $60 = 6 \cdot 10$ to get the number of hours, and by $24 = 3 \cdot 8$ to get the number of days:

$$\frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(5 \cdot 12) \cdot (6 \cdot 10) \cdot (3 \cdot 8)} = 13 \cdot 11 \cdot 9 \cdot 7 \cdot 4 \cdot 2.$$

Note that a year is either 1 or 2 days longer than $364 = 7 \cdot 4 \cdot 13$ days, showing that 13! seconds is a little short of $11 \cdot 9 \cdot 2 = 198$ years. Each leap year has 2 more than 364 days, and other years have 1 more than 364 days. Thus, 13! seconds is short of 198 years by 197 days plus one more day for each leap year between 2022 and $2022 + 197 = 2219$. The leap years in that range are $2024, 2028, 2032, ..., 2216$ except for 2100 and 2200. This accounts for $\frac{2216-2020}{4} - 2 = 47$ leap years. Therefore, the number of years is 197, and the number of days is $364 - (197 + 47) = 120$. The requested sum is $197 + 120 = 317$. □

Problem 1.17. A rectangle with width 30 inches has the property that all points in the rectangle are within 12 inches of at least one of the diagonals of the rectangle. Find the maximum possible length for the rectangle in inches.

Solution. $\boxed{\textbf{Answer: 40}}$
Label the rectangle $ABCD$ with $AB = 30$. Let E be the midpoint of $\overline{AD}$, and F be the perpendicular projection of E onto the diagonal $\overline{AC}$. Let $x = AE$



Because the midpoints of the sides of a rectangle are the points on the rectangle farthest from the diagonals, x is as great as possible when $EF = 12$. Because $\triangle ACD \simeq \triangle AEF$,

$$\frac{AC}{CD} = \frac{AE}{EF} \qquad \text{so} \qquad \frac{\sqrt{30^2 + 4x^2}}{30} = \frac{x}{12},$$

which simplifies to $x = 20$. Thus, the requested side length is $AD = 2x = 40$ inches. Note that on any rectangle, the four midpoints of the sides are all the same distance from the diagonals of the rectangle. □

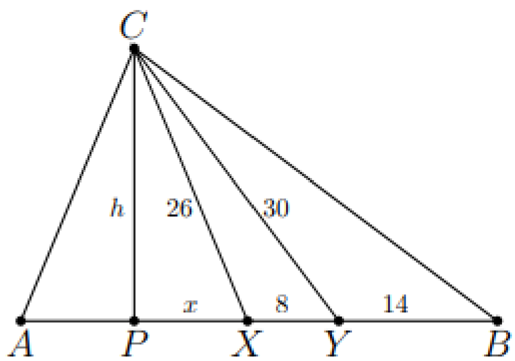
Problem 1.18. Let a and b be positive integers satisfying $3a < b$ and $a^2 + ab + b^2 = (b + 3)^2 + 27$. Find the minimum possible value of $a + b$.

Solution. **Answer: 25**
Squaring the binomial gives $a^2 + ab + b^2 = b^2 + 6b + 9 + 27$ which simplifies to $0 = a^2 + ab - 6b - 36 = (a - 6)(a + b + 6)$. Because a and b are positive, $a + b + 6 > 0$, so it must be that $a = 6$. Because $3a < b$, the least value of b that satisfies the given conditions is $b = 19$. The minimum possible value of $a + b$ is $6 + 19 = 25$ □

Problem 1.19. Points X and Y lie on side $\overline{AB}$ of $\triangle ABC$ such that $AX = 20$, $AY = 28$, and $AB = 42$. Suppose $XC = 26$ and $YC = 30$. Find $AC + BC$.

Solution. **Answer: 66**
Note that $XY = 8$ and $YB = 14$. Let P be the foot of the altitude of $\triangle ABC$ to vertex C. Let $x = PX$ and $h = PC$. Then applying the Pythagorean Theorem to $\triangle PCX$ and $\triangle PCY$ gives

$$x^2 + h^2 = 26^2 \quad \text{and} \quad (x + 8)^2 + h^2 = 30^2.$$



Subtracting the first equation from the second gives $16x + 64 = 30^2 - 26^2$, so $x = 10$ and $h = 24$. It follow that P is the midpoint of $\overline{AX}$, so $AC = 26$ and $BC = \sqrt{(AB - AP)^2 + h^2} = \sqrt{32^2 + 24^2} = 40$. The requested sum is $26 + 40 = 66$. One can also find h by finding the area of $\triangle XCY$ using Heron's Formula and setting that equal to $\frac{8h}{2}$. □

Problem 1.20. There are real numbers x, y, and z such that the value of

$$x + y + z - \left(\frac{x^2}{5} + \frac{y^2}{6} + \frac{z^2}{7}\right)$$

reaches its maximum of $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n + x + y + z$.

Solution. $\boxed{\textbf{Answer: 20}}$
We will complete the squares in the expression to yield

$$\begin{aligned}
&= \left(x - \frac{x^2}{5}\right) + \left(y - \frac{y^2}{6}\right) + \left(z - \frac{z^2}{7}\right) \\
&= \left(\frac{5}{4} + \frac{6}{4} + \frac{7}{4}\right) + \left(-\frac{5}{4} + x - \frac{x^2}{5}\right) + \left(-\frac{6}{4} + y - \frac{y^2}{6}\right) + \left(-\frac{7}{4} + z - \frac{z^2}{7}\right) \\
&= \frac{9}{2} - 5\left(\frac{1}{2} - \frac{x}{5}\right)^2 - 6\left(\frac{1}{2} - \frac{y}{6}\right)^2 - 7\left(\frac{1}{2} - \frac{z}{7}\right)^2.
\end{aligned}$$

Therefore, the maximum value of the expression is $\frac{9}{2}$ obtained when $(x, y, z) = (\frac{5}{2}, 3, \frac{7}{2})$. The requested sum is $9 + 2 + \frac{5}{2} + 3 + \frac{7}{2} = 20$. $\square$